Эквивалентность некоторых квантовомеханических моделей

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Уравнение Шредингера. Уравнение Риккати

$$-\frac{\mathrm{d}^2}{2\mathrm{dx}^2}\psi_0 + (-\mathrm{E} + \mathrm{V})\psi_0 = 0, \quad \mathrm{E} == 0, \quad \psi_0'' = 2\mathrm{V}\psi_0$$

$$\left(\frac{\psi_0'}{\psi_0}\right)' = \frac{\psi_0''}{\psi_0} - \frac{\psi_0'^2}{\psi_0^2}, \quad \left(\frac{\psi_0'}{\psi_0}\right)' + \frac{\psi_0'^2}{\psi_0^2} = 2\mathrm{V}$$

$$\mathrm{f}^2(\mathrm{x}) - \mathrm{f}'(\mathrm{x}) = 2\mathrm{V}(\mathrm{x}), \quad \mathrm{f}(\mathrm{x}) = -\frac{\psi_0'}{\psi_0}$$

Уравнение Риккати

$$\mathrm{f}'(\mathrm{x}) = \mathrm{f}^2(\mathrm{x}) - 2\mathrm{V}(\mathrm{x}), \quad \mathrm{f}(\mathrm{x}) = -\frac{\psi_0'}{\psi_0}$$

Пусть $\psi_0(\mathrm{x}) = (\mathrm{g}'(\mathrm{x}))^{-\frac{1}{2}}$

$$\mathrm{V}(\mathrm{x}) = \frac{1}{2}(\mathrm{f}^2(\mathrm{x}) - \mathrm{f}'(\mathrm{x})) = -\frac{1}{4}\left[\frac{\mathrm{g}'''(\mathrm{x})}{\mathrm{g}'(\mathrm{x})} - \frac{3}{2}\left(\frac{\mathrm{g}''(\mathrm{x})}{\mathrm{g}'(\mathrm{x})}\right)^2\right] = -\frac{1}{4}\mathrm{S_g}(\mathrm{x}),$$

$$\mathrm{x} = \phi(\tilde{\mathrm{x}}), \quad \mathrm{f} = \frac{\mathrm{a}\tilde{\mathrm{f}} + \mathrm{b}}{\mathrm{c}\tilde{\mathrm{f}} + \mathrm{d}}$$

$$\mathrm{S}_{\mathrm{h}(\mathrm{g}(\mathrm{x}))}(\mathrm{x}) = \mathrm{S_h}(\mathrm{g}(\mathrm{x})) * (\mathrm{g}'(\mathrm{x}))^2 + \mathrm{S_g}(\mathrm{x})$$

Преобразования уравнения Риккати. Потенциал Морса

$$y'_x = y^2 - \frac{\alpha^2}{4} + e^{-2\alpha x} f(e^{-\alpha x}),$$

$$\omega'_\xi = \omega^2 + \frac{1}{\alpha^2} f(\xi),$$

$$\xi = e^{-\alpha x}, \quad w = \frac{1}{-\alpha} e^{\alpha x} y - \frac{1}{2} e^{\alpha x}$$

$-2V(x)$	$f(e^{-\alpha x})$	$-2\widetilde{V}(\xi)$
$-\frac{\alpha^2}{4} + D\left[e^{-4\alpha x} - 2e^{-2\alpha x}\right]$	$-2 + e^{-2\alpha x}$	$\frac{D}{\alpha^2}\xi^2 - \frac{2D}{\alpha^2}$
$-\frac{\alpha^2}{4} + D\left[e^{-4\alpha x} - 2e^{-2\alpha x} + C\right]$	$-2 + e^{-2\alpha x} + \frac{1}{e^{-2\alpha x}}$	$\frac{D}{\alpha^2}\xi^2 + \frac{DC}{\alpha^2}\frac{1}{\xi^2} - \frac{2D}{\alpha^2}$
$-\frac{\beta^2}{4} + D\left[e^{-2\beta x} - 2e^{-\beta x}\right]$	$1 + \frac{-\beta}{e^{-\beta x}}$	$\frac{D}{\beta^2} - \frac{2D}{\beta^2}\frac{1}{\lvert\xi\rvert}$

Преобразования уравнения Риккати. Потенциал Морса

$\alpha = 1/2, \quad \mathrm{D} = 1$:

$$\psi_0 = \mathrm{e}^{-\mathrm{e}^{-\mathrm{x}}} 2^{\mathrm{s}} (\mathrm{e}^{-\mathrm{x}})^{\mathrm{s}} \equiv \exp(-\int \mathrm{y}\mathrm{dx})$$

=>

$$\mathrm{e}^{-\xi} 2^{\mathrm{s}} \xi^{\mathrm{s}} \equiv \exp\left[-\int \omega \mathrm{d}\xi - \int \frac{1}{2\xi} \mathrm{d}\xi\right] = \psi_0(\xi) \mathrm{e}^{-\int \frac{1}{2\xi} \mathrm{d}\xi} = \psi_0(\xi) \frac{1}{\sqrt{\xi}}$$

$$\psi_0(\xi) = \mathrm{C} \mathrm{e}^{-\xi} \xi^{\mathrm{s} - \frac{1}{2}}$$

$$f'_x(x) = f^2(x) - 2V(x) \quad => \quad w'_\xi(\xi) = w^2(\xi) - 2\widetilde{V}(\xi)$$

$-2V(x)$	$-2\widetilde{V}(\xi)$	ξ	ω
$-\lambda^2 + a\,ch^n(\lambda x)sh^{-n-4}(\lambda x)$	$\lambda^{-2}\xi^n$	$cth(\lambda x)$	$-\frac{1}{\lambda}sh^2(\lambda x)y - sh(\lambda x)ch(\lambda x)$

$$y'_x = ay^2 + bx^{n-2} + \frac{c}{x^2}, \quad aA^2 - A + c = 0, \quad \omega = xy + A$$

$$x\omega'_x = a\omega^2 + (1 - 2aA)\omega + bx^n$$

$$x\omega'_x = \alpha\omega^2 + \beta\omega + \gamma x^n, \quad \xi = x^\beta, \quad \eta = yx^{-\beta}, \quad m = \frac{n}{\beta} - 2$$

$$\eta'_\xi = \frac{\alpha}{\beta}\eta^2 + \frac{\gamma}{\beta}\xi^m$$

Двойственные потенциалы

$$\mathrm{H} = \begin{pmatrix} \mathrm{H_1} \\ \mathrm{H_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\mathrm{p}^2 + \mathrm{f}^2(\mathrm{x}) + \mathrm{f}'(\mathrm{x})) & 0 \\ 0 & \frac{1}{2}(\mathrm{p}^2 + \mathrm{f}^2(\mathrm{x}) - \mathrm{f}'(\mathrm{x})) \end{pmatrix}$$

Если $\mathrm{E} = 0$

$$\Psi_{\pm} = \mathrm{C}\exp\left(\pm\int_0^{\mathrm{x}} \mathrm{f}(\mathrm{s})\mathrm{ds}\right)$$

$$1)||\Psi_+|| = \infty, \quad ||\Psi_-|| = \infty$$

$$2)||\Psi_+|| = \infty, \quad ||\Psi_-|| < \infty$$

$$3)||\Psi_+|| < \infty, \quad ||\Psi_-|| = \infty$$

Гармонический осциллятор

ψ_0	$f(x) = -\frac{\psi_0'}{\psi_0}$	V_2	V_1
$e^{-x^2/2}$	x	$\frac{1}{2}(x^2 - 1)$	$\frac{1}{2}(x^2 + 1)$
$\sqrt{2}xe^{-x^2/2}$	$x - \frac{1}{x}$	$\frac{1}{2}(x^2 - 3)$	$\frac{1}{2}(x^2 + \frac{2}{x^2} - 1)$
$\frac{1}{\sqrt{2}}(2x^2 - 1)e^{-x^2/2}$	$x - \frac{4x}{2x^2-1}$	$\frac{1}{2}(x^2 - 5)$	$\frac{1}{2}(x^2 + \frac{8}{2x^2-1} + \frac{16}{(2x^2-1)^2} - 3)$
$\frac{(8x^3-12x)}{4\sqrt{3}}e^{-x^2/2}$	$\frac{2x^4-9x^2+3}{2x^3-3x}$	$\frac{1}{2}(x^2 - 7)$	$\frac{1}{2}(x^2 + \frac{8}{2x^2-3} + \frac{48}{(2x^2-3)^2} + \frac{2}{x^2} - 5)$
$e^{x^2/2}$	$-x$	$\frac{1}{2}(x^2 + 1)$	$\frac{1}{2}(x^2 - 1)$
$\sqrt{2}xe^{x^2/2}$	$x + \frac{1}{x}$	$\frac{1}{2}(x^2 + 3)$	$\frac{1}{2}(x^2 + \frac{2}{x^2} + 1)$

Потенциал Калоджеро

$$f(x) = x + \frac{a}{x}$$

$$H = \begin{pmatrix} \frac{1}{2}(p^2 + x^2 + 2a + 1 + \frac{a(a-1)}{x^2}) & 0 \\ 0 & \frac{1}{2}(p^2 + x^2 + 2a - 1 + \frac{a(a+1)}{x^2}) \end{pmatrix}$$

Потенциал Калоджеро

ψ_0	V_2	V_1
$\frac{1}{x}e^{-x^2/2}$	$\frac{1}{2}(\frac{2}{x^2}+x^2+1)$	$\frac{1}{2}(x^2+3)$
$(1-2x^2-\frac{1}{x^2})e^{-\frac{x^2}{2}}$	$x^2+\frac{2}{x^2}-3$	$\frac{1}{2}(x^2+\frac{8}{2x^2+1}-\frac{16}{(2x^2+1)^2}-1)$
$x^2e^{-x^2/2}$	$\frac{1}{2}(x^2+\frac{2}{x^2}-5)$	$\frac{1}{2}(x^2+\frac{6}{x^2}-3)$
$2(x^2-1)x^2e^{-\frac{x^2}{2}}$	$\frac{1}{2}[\frac{(x^6-10x^4+17x^2-2)^2}{x^4(x^2-1)^2}+$ $+\frac{2x(x^8-2x^6-7x^4+4x^2-2)}{x^4(x^2-1)^2}]$	$\frac{1}{2}[\frac{(x^6-10x^4+17x^2-2)^2}{x^4(x^2-1)^2}-$ $-\frac{2x(x^8-2x^6-7x^4+4x^2-2)}{x^4(x^2-1)^2}]$
$(4x^6-22x^4+22x^2)e^{-\frac{x^2}{2}}$	$\frac{1}{2}(x^2+\frac{2}{x^2}+$ $+\frac{2(6x^2-11}{2x^4-11x^2+11)}-13)$	$\frac{1}{2}(x^2+\frac{6}{x^2}+$ $+\frac{264x^2}{(2x^4-11x^2+11)^2)^2}+$ $+\frac{2(14x^2+11)}{2x^4-11x^2+11}-11)$
$x^{-\alpha}e^{-x^2/2}$	$\frac{1}{2}(x^2-2\alpha-\frac{\alpha^2}{x^2})$	$(x+\frac{\alpha}{x})^2+1-\frac{\alpha}{x^2}$

Потенциал Морса

$$V(x) = Ae^{-2\alpha x} + Be^{-\alpha x}$$

$$f(x) = -ae^{-\alpha x} + b$$

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(p^2 + a^2e^{-2\alpha x} - (2ab - a\alpha)e^{-\alpha x} + b^2) & 0 \\ 0 & \frac{1}{2}(p^2 + a^2e^{-2\alpha x} - (2ab + a\alpha)e^{-\alpha x} + b^2) \end{pmatrix}$$

$$E_0 = -A\left(\frac{1}{4}\frac{B^2}{A} + \frac{\alpha}{2\sqrt{A}}\frac{B}{\sqrt{2}} + \frac{\alpha^2}{8}\right)$$

Для $\alpha = 1, \quad A = 1$

$$V_2(x) = s^2 - 2\sqrt{2}se^{-x} - \frac{2se^{-x}}{2s - e^{-x} + 1} + \frac{2\sqrt{2}e^{-2x}}{2s - e^{-x} + 1} - \frac{e^{-x}}{2s - e^{-x} + 1} + 2e^{-2x} - \sqrt{2}e^{-x}$$

$$V_1(x) = s^2 - 2\sqrt{2}se^{-x} - \frac{2se^{-x}}{2s - e^{-x} + 1} + \frac{2\sqrt{2}e^{-2x}}{2s - e^{-x} + 1} + \frac{e^{-x}}{2s - e^{-x} + 1} + \frac{2e^{-2x}}{(2s - e^{-x} + 1)^2} + \\ +2e^{-2x} + \sqrt{2}e^{-x}$$

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